



# Test vectors for trilinear forms: the case of two principal series

Mladen Dimitrov, Louise Nyssen

## ► To cite this version:

Mladen Dimitrov, Louise Nyssen. Test vectors for trilinear forms: the case of two principal series. 2008. hal-00328183

**HAL Id: hal-00328183**

**<https://hal.science/hal-00328183>**

Preprint submitted on 9 Oct 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Test vectors for trilinear forms : the case of two principal series

Mladen Dimitrov and Louise Nyssen

dimitrov@math.jussieu.fr, lnyssen@math.univ-montp2.fr

October 9, 2008

## 1 Introduction

Let  $F$  be a finite extension of  $\mathbb{Q}_p$  with ring of integers  $\mathcal{O}$  and uniformizing parameter  $\pi$ . Let  $V_1$ ,  $V_2$  and  $V_3$  be three irreducible, admissible, infinite dimensional representations of  $G = \mathrm{GL}_2(F)$  of central characters  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  and conductors  $n_1$ ,  $n_2$  and  $n_3$ . Using the theory of Gelfand pairs, Dipendra Prasad proves in [P] that the space of  $G$ -invariant linear forms on  $V_1 \otimes V_2 \otimes V_3$  has dimension at most one and gives a precise criterion for this dimension to be one, that we will now explain.

Let  $D^*$  be the group of invertible elements of the unique quaternion division algebra  $D$  over  $F$ . When  $V_i$  is a discrete series representation of  $G$ , denote by  $V'_i$  the irreducible representation of  $D^*$  associated to  $V_i$  by the Jacquet-Langlands correspondence. Again, by the theory of Gelfand pairs, the space of  $D^*$ -invariant linear forms on  $V'_1 \otimes V'_2 \otimes V'_3$  has dimension at most one.

A necessary condition for the existence on a non-zero  $G$ -invariant linear form on  $V_1 \otimes V_2 \otimes V_3$  (resp. non-zero  $D^*$ -invariant linear form on  $V'_1 \otimes V'_2 \otimes V'_3$ ), that we will *always assume*, is that

$$\omega_1 \omega_2 \omega_3 = 1.$$

Let  $\sigma_i$  be the two dimensional representations of the Weil-Deligne group of  $F$  associated to  $V_i$ . The triple tensor product  $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$  is an eight dimensional symplectic representation of the Weil-Deligne group having a local root number  $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3)$  equal to 1 or  $-1$ . When  $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ , one can prove that the  $V_i$ 's are all discrete series representations of  $G$ .

**Theorem 1.** (*Prasad [P, Theorem 1.4]*) *If all the  $V_i$ 's are supercuspidal, assume that the residue characteristic of  $F$  is not 2. Then*

- $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$  if, and only if, there exists a non-zero  $G$ -invariant linear form on  $V_1 \otimes V_2 \otimes V_3$ , and
- $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$  if, and only if, there exists a non-zero  $D^*$ -invariant linear form on  $V'_1 \otimes V'_2 \otimes V'_3$ .

Given a non zero  $G$ -invariant linear form  $\ell$  on  $V_1 \otimes V_2 \otimes V_3$ , or a non-zero  $D^*$ -invariant linear form  $\ell'$  on  $V'_1 \otimes V'_2 \otimes V'_3$ , the goal is to find a vector in  $V_1 \otimes V_2 \otimes V_3$  which is not in the kernel of  $\ell$ , or a vector in  $V'_1 \otimes V'_2 \otimes V'_3$  which is not in the kernel of  $\ell'$ . Such a vector is called a test vector. The following results of Prasad and Gross-Prasad show that new vectors can sometimes be used as test vectors. In what follows  $v_i$  denotes a new vector in  $V_i$  (see §2.2).

**Theorem 2.** (Prasad [P, Theorem 1.3]) *If all the  $V_i$ 's are unramified principal series, then  $v_1 \otimes v_2 \otimes v_3$  is a test vector.*

**Theorem 3.** (Gross and Prasad [G-P, Proposition 6.3]) *Suppose all the  $V_i$ 's are unramified twists of the Steinberg representation.*

- *If  $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$ , then  $v_1 \otimes v_2 \otimes v_3$  is a test vector.*
- *If  $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$  and if  $R$  is the unique maximal order in  $D$ , then any vector belonging to the unique line in  $V_1' \otimes V_2' \otimes V_3'$  fixed by  $R^* \times R^* \times R^*$  is a test vector.*

Actually, the proof by Gross and Prasad of the first statement of the above theorem contains another result :

**Theorem 4.** *If two of the  $V_i$ 's are unramified twists of the Steinberg representation and the third one is an unramified principal series, then  $v_1 \otimes v_2 \otimes v_3$  is a test vector.*

However, as mentioned in [G-P], new vectors are not always test vectors. Let  $K = \text{GL}(\mathcal{O})$  be the maximal compact subgroup of  $G$  and suppose that  $V_1$  and  $V_2$  are unramified, but  $V_3$  is ramified. Since  $v_1$  and  $v_2$  are  $K$ -invariant and  $\ell$  is  $G$ -equivariant,  $v \mapsto \ell(v_1 \otimes v_2 \otimes v)$  defines a  $K$ -invariant linear form on  $V_3$ . Since  $V_3$  is ramified, so is its contragredient, and therefore the above linear form has to vanish. In particular  $\ell(v_1 \otimes v_2 \otimes v_3) = 0$ .

To go around this obstruction for new vectors to be test vectors, Gross and Prasad made the following suggestion : suppose that  $V_3$  has conductor  $n = n_3 \geq 1$ ; since  $V_3$  has unramified central character, its contragredient representation has non-zero invariant vectors by the  $n$ -th standard Iwahori subgroup  $I_n = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi^n \mathcal{O} & \mathcal{O}^\times \end{pmatrix}$  of  $G$ ; put  $\gamma = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix}$  and let  $v_1^* \in V_1$  be a non-zero vector on the line fixed by the maximal compact subgroup  $\gamma^n K \gamma^{-n}$  of  $G$ ; since  $K \cap \gamma^n K \gamma^{-n} = I_n$ , the linear form on  $V_3$  given by  $v \mapsto \ell(v_1^* \otimes v_2 \otimes v)$  is not necessarily zero and there is still hope for  $v_1^* \otimes v_2 \otimes v_3$  to be a test vector. This is the object of the following theorem

**Theorem 5.** *If  $V_1$  and  $V_2$  are unramified and  $V_3$  has conductor  $n_3$ , then  $v_1^* \otimes v_2 \otimes v_3$  is a test vector, where  $v_1^* = \gamma^{n_3} \cdot v_1$ .*

Theorem 5 for  $n_3 = 1$ , together with Theorems 2, 3 and 4, completes the study of test vectors when the  $V_i$ 's have conductors 0 or 1 and unramified central characters.

Assume from now on that  $V_1$  and  $V_2$  are (ramified or unramified) principal series. Then for  $i = 1, 2$  there exist quasi-characters  $\mu_i$  and  $\mu'_i$  of  $F^\times$  such that  $\mu'_i \mu_i^{-1} \neq |\cdot|^{\pm 1}$ , and

$$V_i = \text{Ind}_B^G \chi_i, \text{ with } \chi_i \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu_i(a) \mu'_i(d).$$

According to Theorem 1 there exists a non-zero  $G$ -invariant linear form  $\ell$  on  $V_1 \otimes V_2 \otimes V_3$ , so we are looking for a test vector in  $V_1 \otimes V_2 \otimes V_3$ . The following theorem is our main result.

**Theorem 6.** *Suppose that  $V_1$  and  $V_2$  are principal series such that  $\mu_1$  and  $\mu'_2$  are unramified. Put*

$$x = \max(n_2 - n_1, n_3 - n_1) \quad \text{and} \quad v_1^* = \gamma^x \cdot v_1.$$

*Then  $x \geq 0$  and, if  $v_1^* \otimes v_2 \otimes v_3$  is not a test vector, then*

- either  $n_1 = 0$ ,  $n_2 = n_3 > 0$  and  $\gamma^{n_2-1} \cdot v_1 \otimes v_2 \otimes v_3$  is a test vector,
- or  $n_2 = 0$ ,  $n_1 = n_3 > 0$  and  $v_1 \otimes \gamma \cdot v_2 \otimes v_3$  is a test vector,
- or  $\widetilde{V}_3$  is a quotient of  $\text{Ind}_B^G(\chi_1 \chi_2 \delta^{\frac{1}{2}})$ ,  $n_1 + n_2 = n_3$  and  $v_1 \otimes \gamma^{n_1} \cdot v_2 \otimes v_3$  is a test vector.

The assumptions of the theorem imply in particular that  $V_1$  and  $V_2$  have minimal conductor among their twists. If  $V_1$  and  $V_2$  are two arbitrary principal series, then one can always find characters  $\eta_1$ ,  $\eta_2$  and  $\eta_3$  of  $F^\times$  with  $\eta_1 \eta_2 \eta_3 = 1$ , such that the above theorem applies to  $(V_1 \otimes \eta_1) \otimes (V_2 \otimes \eta_2) \otimes (V_3 \otimes \eta_3)$ . Nevertheless, we found also interesting to study the case when  $\mu_1$  or  $\mu'_2$  is ramified. Then we are able to show that certain new vectors are *not* test vectors, while *a priori* this cannot be seen by a direct argument (the obstruction of Gross and Prasad described above does not apply to this case). Put  $m_1 = \text{cond}(\mu'_1)$  and  $m_2 = \text{cond}(\mu'_2)$

**Theorem 7.** *Suppose that  $\mu_1$  or  $\mu'_2$  is ramified. Let  $x$ ,  $y$  and  $z$  be integers such that*

- $x \geq m_1$ ,
- $y \geq m_2$ ,
- $x - n_3 \geq z \geq y$ , and
- $x - y \geq \max(n_1 - m_1, n_2 - m_2, 1)$ .

Put

$$\begin{aligned} v_1^* &= \begin{cases} \gamma^{x-m_1} \cdot v_1 & , \text{ if } \mu'_1 \text{ is ramified,} \\ \gamma^x \cdot v_1 - \beta_1 \gamma^{x-1} \cdot v_1 & , \text{ if } \mu'_1 \text{ is unramified.} \end{cases} \\ v_2^* &= \begin{cases} \gamma^{y-m_2} \cdot v_2 & , \text{ if } \mu_2 \text{ is ramified.} \\ \gamma^{y-n_2} \cdot v_2 - \alpha_2^{-1} \gamma^{y-n_2+1} \cdot v_2 & , \text{ if } \mu_2 \text{ is unramified.} \end{cases} \end{aligned} \tag{1}$$

Then

$$\ell(v_1^* \otimes v_2^* \otimes \gamma^z \cdot v_3) = 0.$$

We will prove theorems 6 and 7 by following the pattern of the proof of Theorem 2 in [P], with the necessary changes.

We believe that suitable generalization of the method of Gross and Prasad would give test vectors in the case where at least two of the  $V_i$ 's are special representations, as well as in the case where one is a special representation and one is a principal series. On the other hand in order to find test vectors in the case where at least two of the  $V_i$ 's are supercuspidal, one should use different techniques, involving probably computations in Kirillov models.

The search for test vectors in our setting is motivated by subconvexity problems for  $L$ -functions of triple products of automorphic forms on  $\text{GL}(2)$ . Roughly speaking, one wants to bound the value of the  $L$ -function along the critical line  $\Re(z) = \frac{1}{2}$ . In [B-R 1] and [B-R 2] Joseph Bernstein and Andre Reznikov establish a *subconvexity bound* when the *eigenvalue* attached to one of the representations varies. Philippe Michel and Akshay Venkatesh considered the case when the *level* of one representation varies. More details about subconvexity and those related techniques can be found in [V] or [M-V]. Test vectors are key ingredients.

Bernstein and Reznikov use an explicit test vector. Venkatesh uses a theoretical one, but explains that the bounds would be better with an explicit one (see [V, §5]).

There is an extension of Prasad's result in [H-S], where Harris and Scholl prove that the dimension of the space of  $G$ -invariant linear forms on  $V_1 \otimes V_2 \otimes V_3$  is one when  $V_1$ ,  $V_2$  and  $V_3$  are principal series representations (either irreducible or reducible, but with infinite dimensional irreducible subspace). They apply their result to the global setting to construct elements in the motivic cohomology of the product of two modular curves predicted by Beilinson.

## Acknowledgments.

We would like to thank Philippe Michel for suggesting the study of this problem, and of course Benedict Gross and Diprenda Prasad for their articles full of inspiration. The second named author would like to thank also Paul Broussous and Nicolas Templier for many interesting discussions, and Wen-Ching Winnie Li for the opportunity to spend one semester at PennState University where the first draft of this paper was written.

## 2 Background on induced admissible representations of $GL(2)$ .

### 2.1 About induced and contragredient representations.

Let  $(\rho, W)$  be a smooth representation of a closed subgroup  $H$  of  $G$ . Let  $\Delta_H$  be the modular function on  $H$ . The induction of  $\rho$  from  $H$  to  $G$ , denoted  $\text{Ind}_H^G \rho$ , is the space of functions  $f$  from  $G$  to  $W$  satisfying the two following conditions :

- (1)  $\forall h \in H, \quad \forall g \in G, \quad f(hg) = \Delta_H(h)^{-\frac{1}{2}} \rho(h) f(g),$
- (2) there exists an open compact subgroup  $K_f$  of  $G$  such that

$$\forall k \in K_f, \quad \forall g \in G, \quad f(gk) = f(g)$$

where  $G$  acts by right translation as follows :

$$\forall g, g' \in G, (g \cdot f)(g') = f(g'g).$$

With the additional condition that  $f$  must be compactly supported modulo  $H$ , one gets the *compact* induction denoted by  $\text{ind}_H^G$ . When  $G/H$  is compact, there is no difference between  $\text{Ind}_H^G$  and  $\text{ind}_H^G$ .

Let  $B$  the Borel subgroup of upper triangular matrices in  $G$ , and let  $T$  be the diagonal torus. The character  $\Delta_T$  is trivial and we will use  $\Delta_B = \delta^{-1}$  with  $\delta \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |\frac{a}{d}|$  where  $|\cdot|$  is the normalised valuation of  $F$ . The quotient  $B \backslash G$  is compact and can be identified with  $\mathbb{P}^1(F)$ .

For a smooth representation  $V$  of  $G$ , the contragredient representation  $\tilde{V}$  is the space of smooth linear forms  $l$  on  $V$ , where  $G$  acts as follows :

$$\forall g \in G, \quad \forall v \in V, \quad (g \cdot l)(v) = l(g^{-1} \cdot v).$$

We refer the reader to [B-Z] for more details about induced and contragredient representations.

## 2.2 New vectors and ramification.

Let  $V$  be an irreducible, admissible, infinite dimensional representation of  $G$  with central character  $\omega$ . Then  $\tilde{V} \cong V \otimes \omega^{-1}$ . To the descending chain of compact subgroups of  $G$

$$K = I_0 \supset I_1 \supset \cdots \supset I_n \supset I_{n+1} \cdots$$

one can associate an ascending chain of vector spaces

$$V^{I_0, \omega} = V^K, \text{ and for all } n \geq 1, \quad V^{I_n, \omega} = \left\{ v \in V \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot v = \omega(d)v, \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_n \right\}.$$

There exists a minimal  $n$  such that the vector space  $V^{I_n, \omega}$  is non-zero. It is necessarily one dimensional and any non-zero vector in it is called a *new vector* of  $V$ . The integer  $n$  is the *conductor* of  $V$ . The representation  $V$  is said to be *unramified* if  $n = 0$ .

More information about new vectors can be found in [C].

## 2.3 New vectors as functions on $G$ .

Let  $V$  be a principal series of  $G$ , with central character  $\omega$ , and conductor  $n$ . There exist quasi-characters  $\mu$  and  $\mu'$  of  $F^\times$  such that  $\mu'\mu^{-1} \neq |\cdot|^{\pm 1}$ , and

$$V = \text{Ind}_B^G(\chi) \quad \text{with} \quad \chi \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} = \mu(a)\mu'(d).$$

Then  $\omega = \mu\mu'$  and  $n = \text{cond}(\mu) + \text{cond}(\mu')$ . A new vector  $v$  in  $V$  is a non-zero function from  $G$  to  $\mathbb{C}$  such that for all  $b \in B$ ,  $g \in G$  and  $k = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in I_n$

$$v(bgk) = \chi(b)\delta(b)^{\frac{1}{2}}\omega(d)v(g).$$

Put

$$\alpha^{-1} = \mu(\pi)|\pi|^{\frac{1}{2}} \quad \text{and} \quad \beta^{-1} = \mu'(\pi)|\pi|^{-\frac{1}{2}}.$$

First, we assume that  $V$  is unramified, and we normalise  $v$  so that  $v(1) = 1$ .

**Lemma 2.1.** *If  $V$  is unramified then for all  $r \in \mathbb{N}$ ,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \beta^r & , \text{ if } k \in K \setminus I, \\ \alpha^s \beta^{r-s} & , \text{ if } k \in I_s \setminus I_{s+1} \text{ for } 1 \leq s \leq r-1, \\ \alpha^r & , \text{ if } k \in I_r. \end{cases}$$

Similarly,

$$(\gamma^r \cdot v - \alpha^{-1} \gamma^{r+1} \cdot v)(k) = \begin{cases} \alpha^s \beta^{r-s} - \alpha^{s-1} \beta^{r+1-s} & , \text{ if } k \in I_s \setminus I_{s+1} \text{ for } 0 \leq s \leq r, \\ 0 & , \text{ if } k \in I_{r+1}. \end{cases}$$

Finally, for  $r \geq 1$ ,

$$(\gamma^r \cdot v - \beta \gamma^{r-1} \cdot v)(k) = \begin{cases} \alpha^r (1 - \frac{\beta}{\alpha}) & , \text{ if } k \in I_r, \\ 0 & , \text{ if } k \in K \setminus I_r. \end{cases}$$

*Proof :* If  $k \in I_r$ , then  $\gamma^{-r}k\gamma^r \in K$ , so

$$(\gamma^r \cdot v)(k) = \alpha^r v(\gamma^{-r}k\gamma^r) = \alpha^r.$$

Suppose that  $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_s \setminus I_{s+1}$  for some  $0 \leq s \leq r-1$  (recall that  $I_0 = K$ ). Then  $\pi^{-s}c \in \mathcal{O}^\times$  and

$$(\gamma^r \cdot v)(k) = \alpha^r v \begin{pmatrix} a & \pi^r b \\ \pi^{-r}c & d \end{pmatrix} = \alpha^r v \begin{pmatrix} (ad - bc)\pi^{r-s} & a \\ 0 & \pi^{-r}c \end{pmatrix} = \alpha^s \beta^{r-s}.$$

The second part of the lemma follows by a direct computation.  $\square$

For the rest of this section we assume that  $V$  is ramified, that is  $n \geq 1$ . We put

$$m = \text{cond}(\mu') \quad \text{so that} \quad n - m = \text{cond}(\mu).$$

By Casselman [C, pp.305-306] the restriction to  $K$  of a new vector  $v$  is supported by the double coset of  $\begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix}$  modulo  $I_n$ . In particular if  $\mu'$  is unramified ( $m = 0$ ), then  $v$  is supported by

$$I_n \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} I_n = I_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I_n = K \setminus I.$$

If  $1 \leq m \leq n-1$ , then  $v$  is supported by

$$I_n \begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix} I_n = I_m \setminus I_{m+1}.$$

If  $\mu$  is unramified ( $m = n$ ), then  $v$  is supported by  $I_n$ . We normalise  $v$  so that

$$v \begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix} = 1.$$

**Lemma 2.2.** *If  $\mu$  and  $\mu'$  are both ramified ( $0 < m < n$ ), then for all  $r \in \mathbb{N}$  and  $k \in K$ ,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^r \mu \left( \frac{\det k}{\pi^{-(m+r)}c} \right) \mu'(d) & , \text{ if } k = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in I_{m+r} \setminus I_{m+r+1}, \\ 0 & , \text{ otherwise.} \end{cases}$$

*Proof :* For  $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$  we have

$$\alpha^{-r}(\gamma^r \cdot v)(k) = v(\gamma^{-r}k\gamma^r) = v \begin{pmatrix} a & \pi^r b \\ \pi^{-r}c & d \end{pmatrix}.$$

It is easy to check that for every  $s \geq 1$ ,

$$K \cap B\gamma^r I_s \gamma^{-r} = I_{s+r}.$$

It follows that  $\gamma^r \cdot v$  has its support in  $I_{m+r} \setminus I_{m+r+1}$ . If  $k \in I_{m+r} \setminus I_{m+r+1}$  then  $c \in \pi^{m+r} \mathcal{O}^\times$ ,  $d \in \mathcal{O}^\times$  and we have the following decomposition :

$$\begin{pmatrix} a & \pi^r b \\ \pi^{-r} c & d \end{pmatrix} = \begin{pmatrix} \det k & \pi^{-m} c b \\ 0 & \pi^{-m-r} c d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & \pi^{m+r} c^{-1} \end{pmatrix}. \quad (2)$$

Hence

$$\alpha^{-r}(\gamma^r \cdot v)(k) = \mu(\det(k)) \mu'(\pi^{-m-r} c d) (\mu \mu')(\pi^{m+r} c^{-1}) = \mu\left(\frac{\det(k)}{\pi^{-(m+r)} c}\right) \mu'(d).$$

□

Similarly we obtain :

**Lemma 2.3.** *Suppose that  $\mu$  is unramified and  $\mu'$  is ramified. Then, for all  $r \in \mathbb{N}$  and  $k \in K$ ,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^r \mu'(d) & , \text{ if } k = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in I_{n+r}, \\ 0 & , \text{ otherwise.} \end{cases}$$

$$(\gamma^r \cdot v - \alpha^{-1} \gamma^{r+1} \cdot v)(k) = \begin{cases} \alpha^r \mu'(d) & , \text{ if } k = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in I_{n+r} \setminus I_{n+r+1}, \\ 0 & , \text{ otherwise.} \end{cases}$$

**Lemma 2.4.** *Suppose that  $\mu'$  is unramified and  $\mu$  is ramified. Then for all  $r \in \mathbb{N}$ ,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^s \beta^{r-s} \mu\left(\frac{\det(k)}{\pi^{-s} c}\right) & , \text{ if } k = \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in I_s \setminus I_{s+1}, \text{ with } 0 \leq s \leq r, \\ 0 & , \text{ if } k \in I_{r+1}. \end{cases}$$

Moreover, if  $r \geq 1$ , then

$$(\gamma^r \cdot v - \beta \gamma^{r-1} \cdot v)(k) = \begin{cases} \alpha^r \mu\left(\frac{\det(k)}{\pi^{-r} c}\right) & , \text{ if } k = \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in I_r \setminus I_{r+1}, \\ 0 & , \text{ otherwise.} \end{cases}$$

*Proof :* We follow the pattern of proof of lemma 2.2. The restriction of  $\gamma^r \cdot v$  to  $K$  is zero outside

$$K \cap B \gamma^r (K \setminus I) \gamma^{-r} = K \setminus I_{r+1}.$$

For  $0 \leq s \leq r$  and  $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_s \setminus I_{s+1}$  we use the following decomposition :

$$\begin{pmatrix} a & \pi^r b \\ \pi^{-r} c & d \end{pmatrix} = \begin{pmatrix} -\frac{\det k}{\pi^{-r} c} & a + \frac{\det k}{\pi^{-r} c} \\ 0 & \pi^{-r} c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 + \frac{d}{\pi^{-r} c} \\ 0 & -1 \end{pmatrix}. \quad (3)$$

Since  $d \in \mathcal{O}$  and  $\pi^r c^{-1} \in \mathcal{O}$  we deduce that :

$$\alpha^{-r}(\gamma^r \cdot v)(k) = \mu\left(\frac{\det k}{\pi^{-r} c}\right) \mu'(-\pi^{-r} c) |\pi^r c^{-1}| = \mu\left(\frac{\det k}{\pi^{-s} c}\right) \alpha^{s-r} \beta^{r-s}.$$

□

As direct consequence of these lemmas we obtain



**Lemma 2.5.** *Let  $v_1^*$  and  $v_2^*$  be as in Theorem 7. Then the support of  $v_1^*$  is*

$$\begin{cases} I_x \backslash I_{x+1} & , \text{ if } \mu_1 \text{ is ramified,} \\ I_x & , \text{ if } \mu_1 \text{ is unramified,} \end{cases}$$

*and the support of  $v_2^*$  is*

$$\begin{cases} I_y \backslash I_{y+1} & , \text{ if } \mu'_2 \text{ is ramified,} \\ K \backslash I_{y+1} & , \text{ if } \mu'_2 \text{ is unramified.} \end{cases}$$

### 3 Going down Prasad's exact sequence.

In this section we will explain how Prasad finds a non-zero  $\ell \in \text{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C})$  in the case where  $V_1$  and  $V_2$  are principal series representations.

#### 3.1 Prasad's exact sequence.

The space  $\text{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C})$  is canonically isomorphic to  $\text{Hom}_G(V_1 \otimes V_2, \widetilde{V}_3)$ , hence finding  $\ell$  it is the same as finding a non-zero element  $\Psi$  in it. We have

$$V_1 \otimes V_2 = \text{Res}_G \text{Ind}_{B \times B}^{G \times G} (\chi_1 \times \chi_2)$$

where the restriction is taken with respect to the diagonal embedding of  $G$  in  $G \times G$ . The action of  $G$  on  $(B \times B) \backslash (G \times G) \cong \mathbb{P}^1(F) \times \mathbb{P}^1(F)$  has precisely two orbits.

The first is the diagonal  $\Delta_{B \backslash G}$ , which is closed and can be identified with  $B \backslash G$ . The second is its complement which is open and can be identified with  $T \backslash G$  via the bijection :

$$\begin{aligned} T \backslash G & \longrightarrow (B \backslash G \times B \backslash G) \setminus \Delta_{B \backslash G} \\ Tg & \longmapsto \left( Bg, B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \right) \end{aligned}$$

Hence, there is a short exact sequence of  $G$ -modules :

$$0 \rightarrow \text{ind}_T^G (\chi_1 \chi_2') \xrightarrow{\text{ext}} V_1 \otimes V_2 \xrightarrow{\text{res}} \text{Ind}_B^G (\chi_1 \chi_2 \delta^{\frac{1}{2}}) \rightarrow 0, \quad (4)$$

where  $\chi_2' \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu_2'(a) \mu_2(d)$ . The surjection **res** is given by the restriction to the diagonal.

The injection **ext** takes a function  $f \in \text{ind}_T^G (\chi_1 \chi_2')$  to a function  $F \in \text{Ind}_{B \times B}^{G \times G} (\chi_1 \times \chi_2)$  vanishing on  $\Delta_{B \backslash G}$ , such that for all  $g \in G$

$$F\left(g, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g\right) = f(g).$$

Applying the functor  $\text{Hom}_G(\bullet, \widetilde{V}_3)$  yields a long exact sequence :

$$\begin{aligned} 0 \rightarrow \text{Hom}_G\left(\text{Ind}_B^G (\chi_1 \chi_2 \delta^{\frac{1}{2}}), \widetilde{V}_3\right) & \rightarrow \text{Hom}_G(V_1 \otimes V_2, \widetilde{V}_3) \rightarrow \text{Hom}_G\left(\text{ind}_T^G (\chi_1 \chi_2'), \widetilde{V}_3\right) \\ & \downarrow \\ & \cdots \leftarrow \text{Ext}_G^1\left(\text{Ind}_B^G (\chi_1 \chi_2 \delta^{\frac{1}{2}}), \widetilde{V}_3\right) \end{aligned} \quad (5)$$

### 3.2 The simple case.

The situation is easier if  $V_3$  occurs in  $\text{Ind}_B^G(\chi_1^{-1}\chi_2^{-1}\delta^{-\frac{1}{2}})$ . Then  $\chi_1\chi_2$  does not factor through the determinant and there is a natural surjection

$$\text{Ind}_B^G(\chi_1\chi_2\delta^{\frac{1}{2}}) \twoheadrightarrow \widetilde{V}_3.$$

This surjection is an isomorphism, unless there exists a quasi-character  $\eta$  of  $F^\times$  such that  $\chi_1\chi_2\delta = \eta \circ \det$  in which case the kernel is a line generated by the function  $\eta \circ \det$ . From (4) we obtain a surjective homomorphism  $\Psi$  completing the following commutative diagram :

$$\begin{array}{ccc} V_1 \otimes V_2 & \xrightarrow{\text{res}} & \text{Ind}_B^G(\chi_1\chi_2\delta^{\frac{1}{2}}) \\ \Psi \searrow & & \swarrow \\ & \widetilde{V}_3 & \end{array} \quad (6)$$

Finding a test vector is then reduced to finding an element of  $V_1 \otimes V_2$  whose image by **res** is not zero (resp. not a multiple of  $\eta \circ \det$ ), if  $V_3$  is principal series (resp. special representation).

Following the notations of paragraph 2.3 put, for  $i = 1$  and  $i = 2$

$$m_i = \text{cond}(\mu'_i) \quad \alpha_i^{-1} = \mu_i(\pi)|\pi|^{\frac{1}{2}} \quad \text{and} \quad \beta_i^{-1} = \mu'_i(\pi)|\pi|^{-\frac{1}{2}}.$$

#### 3.2.1 Proof of theorem 7 in the simple case.

To prove theorem 7, suppose that  $\mu_1$  or  $\mu'_2$  is ramified. By our assumptions  $x > y$ , hence  $I_x \cap (K \setminus I_{y+1}) = \emptyset$ . Therefore the supports of  $v_1^*$  and  $v_2^*$  are disjoint and

$$\text{res}(v_1^* \otimes v_2^*) = 0.$$

Using the diagram (6) we see that for any  $v \in V_3$  :

$$\ell(v_1^* \otimes v_2^* \otimes v) = \Psi(v_1^* \otimes v_2^*)(v) = 0.$$

In particular  $\ell(v_1^* \otimes v_2^* \otimes \gamma^z \cdot v_3) = 0$  which proves Theorem 7 in the simple case.

The rest of section 3.2 will be devoted to the proof of Theorems 5 and 6 in the simple case. Consequently, we will suppose that  $\mu_1$  and  $\mu'_2$  are unramified, that is  $m_1 - n_1 = m_2 = 0$ .

#### 3.2.2 Proof of Theorem 5 in the simple case.

Since  $V_1$  and  $V_2$  are unramified, by theorem 2 we may assume that  $V_3$  is ramified. Then necessarily

$$\widetilde{V}_3 = \eta \otimes \text{St},$$

where  $\text{St}$  is the Steinberg representation and  $\eta$  is an unramified character. Hence  $n_3 = 1$  and we will prove that  $\gamma \cdot v_1 \otimes v_2 \otimes v_3$  is a test vector.

The function

$$\begin{cases} G & \longrightarrow \mathbb{C} \\ g & \longmapsto \eta(\det(g))^{-1} \text{res}(\gamma \cdot v_1 \otimes v_2)(g) \end{cases}$$

is not constant, since according to lemma 2.1

$$\eta\left(\det(1)\right)^{-1}(\gamma \cdot v_1 \otimes v_2)(1) = v_1(\gamma)v_2(1) = \alpha_1$$

and

$$\eta\left(\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)^{-1}(\gamma \cdot v_1 \otimes v_2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \eta(-1)v_1\begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix} = \beta_1,$$

and  $\alpha_1 \neq \beta_1$  because  $V_1$  is a principal series.

Hence  $\Psi(\gamma \cdot v_1 \otimes v_2) \neq 0$ . Moreover, since

$$\gamma K \gamma^{-1} \cap K = I$$

we deduce that

$$\Psi(\gamma \cdot v_1 \otimes v_2) \in \widetilde{V}_3^{I, \omega_3^{-1}}.$$

Hence  $\Psi(\gamma \cdot v_1 \otimes v_2)$  cannot vanish on the line  $V_3^{I, \omega_3}$ , which is generated by  $v_3$ , and therefore  $\gamma \cdot v_1 \otimes v_2 \otimes v_3$  is a test vector.

This completes the proof of Theorem 5 in the simple case.

### 3.2.3 Proof of Theorem 6 in the simple case, when $\widetilde{V}_3$ is a special representation.

Assume now that

$$\widetilde{V}_3 = \eta \otimes \text{St},$$

where St is the Steinberg representation and  $\eta$  is a character. Since

$$\eta = \mu_1 \mu_2 |\cdot| = \mu'_1 \mu'_2 |\cdot|^{-1}$$

and  $\mu_1$  and  $\mu'_2$  are unramified, it follows that  $\eta$  is unramified if, and only if, both  $V_1$  and  $V_2$  are unramified. Since this case was taken care of in the previous paragraph, we can assume for the rest of this paragraph that  $\eta$  is ramified. Then

$$n_1 = n_2 = \text{cond}(\eta) \geq 1 \quad \text{and} \quad n_3 = 2n_1 = n_1 + n_2.$$

We will prove that  $v_1 \otimes \gamma^{n_1} \cdot v_2 \otimes v_3$  is a test vector.

The function

$$\begin{cases} G & \longrightarrow \mathbb{C} \\ g & \longmapsto \eta\left(\det(g)\right)^{-1} \text{res}(v_1 \otimes \gamma^{n_1} \cdot v_2)(g) \end{cases}$$

is not constant, since according to lemmas 2.3 and 2.4

$$\eta\left(\det(1)\right)^{-1}(v_1 \otimes \gamma^{n_1} \cdot v_2)(1) = 0$$

whereas

$$\eta\left(\det\begin{pmatrix} 1 & 0 \\ \pi^{n_1} & 1 \end{pmatrix}\right)^{-1}(v_1 \otimes \gamma^{n_1} \cdot v_2)\begin{pmatrix} 1 & 0 \\ \pi^{n_1} & 1 \end{pmatrix} = \alpha_2^{n_1} \neq 0.$$

Hence  $\Psi(v_1 \otimes \gamma^{n_1} \cdot v_2) \neq 0$ . Moreover, since

$$I_{n_1} \cap \gamma^{n_1} I_{n_2} \gamma^{-n_1} = I_{n_1+n_2} = I_{n_3}$$

we deduce that

$$\Psi(v_1 \otimes \gamma^{n_1} \cdot v_2) \in \widetilde{V}_3^{I_{n_3}, \omega_3^{-1}}.$$

Hence  $\Psi(v_1 \otimes \gamma^{n_1} \cdot v_2)$  cannot vanish on the line  $V_3^{I_{n_3}, \omega_3}$ , which is generated by  $v_3$ , and therefore  $v_1 \otimes \gamma^{n_1} \cdot v_2 \otimes v_3$  is a test vector.

### 3.2.4 Proof of Theorem 6 in the simple case, when $\widetilde{V}_3$ is a principal series.

Finally, we consider the case where  $\widetilde{V}_3$  is a principal series representation. Then

$$\widetilde{V}_3 = \text{Ind}_B^G(\chi_1 \chi_2 \delta^{\frac{1}{2}})$$

and

$$n_3 = \text{cond}(\mu_1 \mu_2) + \text{cond}(\mu'_1 \mu'_2) = n_2 + n_1.$$

We will prove that  $v_1 \otimes \gamma^{n_1} \cdot v_2 \otimes v_3$  is a test vector.

According to lemmas 2.1, 2.3 and 2.4 we have

$$(v_1 \otimes \gamma^{n_1} \cdot v_2) \begin{pmatrix} 1 & 0 \\ \pi^{n_1} & 1 \end{pmatrix} = \alpha_2^{n_1} \neq 0,$$

hence  $\text{res}(v_1 \otimes \gamma^{n_1} \cdot v_2) \neq 0$ .

Therefore  $\Psi(v_1 \otimes \gamma^{n_1} v_2) \neq 0$ . Moreover, since

$$I_{n_1} \cap \gamma^{n_1} I_{n_2} \gamma^{-n_1} = I_{n_1+n_2} = I_{n_3}$$

we deduce that

$$\Psi(v_1 \otimes \gamma^{n_1} v_2) \in (\widetilde{V}_3)^{I_{n_3}, \omega_3^{-1}}.$$

Hence  $\Psi(v_1 \otimes \gamma^{n_1} v_2)$  cannot vanish on the line  $V_3^{I_{n_3}, \omega_3}$ , which is generated by  $v_3$ . Thus  $v_1 \otimes \gamma^{n_1} \cdot v_2 \otimes v_3$  is a test vector.

This completes the proof of Theorem 6 in the simple case.

### 3.3 The other case.

The situation is more complicated if  $\text{Hom}_G(\text{Ind}_B^G(\chi_1 \chi_2 \delta^{\frac{1}{2}}), \widetilde{V}_3) = 0$ . By [P, Corollary 5.9] we have  $\text{Ext}_G^1(\text{Ind}_B^G(\chi_1 \chi_2 \delta^{\frac{1}{2}}), \widetilde{V}_3) = 0$ , hence the long exact sequence (5) yields the following isomorphism :

$$\text{Hom}_G(V_1 \otimes V_2, \widetilde{V}_3) \simeq \text{Hom}_G(\text{ind}_T^G(\chi_1 \chi'_2), \widetilde{V}_3).$$

Finally, by Frobenius reciprocity

$$\text{Hom}_G(\text{ind}_T^G(\chi_1 \chi'_2), \widetilde{V}_3) \simeq \text{Hom}_T(\chi_1 \chi'_2, \widetilde{V}_{3|T}).$$

By [W, Lemmes 8-9] the latter space is one dimensional, since the restriction of  $\chi_1 \chi'_2$  to the center equals  $\omega_3^{-1}$  (recall that  $\omega_1 \omega_2 \omega_3 = 1$ ). Thus, we have four canonically isomorphic lines with corresponding bases :

$$\begin{aligned} 0 \neq \ell & \in \text{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C}) \\ & \downarrow \wr \\ 0 \neq \Psi & \in \text{Hom}_G(V_1 \otimes V_2, \widetilde{V}_3) \\ & \downarrow \wr \\ 0 \neq \Phi & \in \text{Hom}_G(\text{ind}_T^G(\chi_1 \chi'_2), \widetilde{V}_3) \\ & \downarrow \wr \\ 0 \neq \varphi & \in \text{Hom}_T(\chi_1 \chi'_2, \widetilde{V}_{3|T}) \end{aligned} \tag{7}$$

Observe that  $\varphi$  can be seen as a linear form on  $V_3$  satisfying :

$$\forall t \in T, \quad \forall v \in V_3, \quad \varphi(t \cdot v) = (\chi_1 \chi_2')(t)^{-1} \varphi(v). \quad (8)$$

**Lemma 3.1.**  $\varphi(v_3) \neq 0$  if, and only if,  $\mu_1 \mu_2'$  is unramified.

*Proof :* Suppose  $\varphi(v_3) \neq 0$ . Since  $v_3 \in V_3$  is a new vector, for all  $a, d \in \mathcal{O}^\times$  we have

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot v_3 = \omega_3(d) v_3 = (\mu_1 \mu_1' \mu_2 \mu_2')(d)^{-1} v_3.$$

Comparing it with (8) forces  $\mu_1 \mu_2'$  to be unramified.

Conversely, assume that  $\mu_1 \mu_2'$  is unramified. Take any  $v \in V_3$  such that  $\varphi(v) \neq 0$ . By smoothness  $v$  is fixed by the principal congruence subgroup  $\ker(K \rightarrow \mathrm{GL}_2(\mathcal{O}/\pi^s))$ , for some  $s \geq 0$ . Then  $\varphi(\gamma^s \cdot v) = (\mu_1 \mu_2')(\pi^s) \varphi(v) \neq 0$  and  $\gamma^s \cdot v$  is fixed by the congruence subgroup

$$I_{2s}^1 := \left\{ k \in K \mid k \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\pi^{2s}} \right\}.$$

By replacing  $\gamma^s \cdot v$  by  $v$  and  $2s$  by  $s$ , we may assume that  $v \in V_3^{I_s^1}$  for some  $s \geq 0$ . Since  $I_s/I_s^1$  is a finite abelian group,  $V_3^{I_s^1}$  decomposes as a direct sum of spaces indexed by the characters of  $I_s/I_s^1$ . Then  $\varphi$  has to be non-zero on  $V_3^{I_s, \omega_3}$  (defined in paragraph 2.2) since by (8),  $\varphi$  vanishes on all other summands of  $V_3^{I_s^1}$ .

By Casselman [C, Theorem 1] the space  $V_3^{I_s, \omega_3}$  has dimension  $n_3 - s + 1$  and has a basis

$$\left( v_3, \quad \gamma \cdot v_3, \quad \dots, \quad \gamma^{n_3-s} \cdot v_3 \right)$$

(recall that  $n_3$  denotes the conductor of  $V_3$ ). Again by (8),  $\varphi(\gamma^i v_3) \neq 0$  for some  $i$  is equivalent to  $\varphi(v_3) \neq 0$ .  $\square$

Notice that, when  $\mu_1 \mu_2'$  and  $\mu_1' \mu_2$  are both unramified, the claim follows from the first case in [G-P, Proposition 2.6] applied to the split torus  $T$  of  $G$ .

## 4 Going up Prasad's exact sequence.

In this section we take as a starting point lemma 3.1 and follow the isomorphisms (7).

### 4.1 From $\varphi$ to $\Phi$ .

Let  $x, y$  and  $z$  be integers such that

$$x - n_3 \geq z \geq y \geq 0 \quad \text{and} \quad x - y \geq 1.$$

For the proof of Theorem 6 we will take

$$x = \max(n_1, n_3) \geq 1 \quad \text{and} \quad y = z = 0.$$

Given a quasi-character  $\mu$  of  $F^\times$  define :

$$\mathcal{O}^\mu = \begin{cases} \mathcal{O} & , \text{ if } \mu \text{ is unramified,} \\ \mathcal{O}^\times & , \text{ if } \mu \text{ is ramified.} \end{cases}$$

Put

$$I_f = \begin{pmatrix} 1 & \pi^{-y}\mathcal{O}^{\mu'_2} \\ \pi^x\mathcal{O}^{\mu_1} & 1 \end{pmatrix},$$

and consider the unique function  $f \in \text{ind}_T^G(\chi_1\chi'_2)$  which is zero outside the open compact subset  $TI_f$  of  $T \backslash G$  and such that for all  $b_0 \in \pi^{-y}\mathcal{O}^{\mu'_2}$  and  $c_0 \in \pi^x\mathcal{O}^{\mu_1}$  we have :

$$f \begin{pmatrix} 1 & b_0 \\ c_0 & 1 \end{pmatrix} = \begin{cases} \mu_1(\frac{\pi^x}{c_0})\mu'_2(b_0\pi^y) & , \text{ if } \mu_1 \text{ and } \mu'_2 \text{ are ramified ;} \\ \mu'_2(b_0\pi^y) & , \text{ if } \mu_1 \text{ is unramified and } \mu'_2 \text{ is ramified ;} \\ \mu_1(\frac{\pi^x}{c_0}) & , \text{ if } \mu_1 \text{ is ramified and } \mu'_2 \text{ is unramified ;} \\ 1 & , \text{ if } \mu_1 \text{ and } \mu'_2 \text{ are unramified.} \end{cases} \quad (9)$$

Since  $x - n_3 \geq z \geq y \geq 0$  and  $x - y \geq 1$  we have

$$I_f \subset \gamma^z I_{n_3}^1 \gamma^{-z}$$

and so every  $k_0 \in I_f$  fixes  $\gamma^z \cdot v_3$ .

By definition, the function  $g \mapsto f(g)\varphi(g\gamma^z \cdot v_3)$  on  $G$  factors through  $T \backslash G$  and

$$(\Phi(f))(\gamma^z \cdot v_3) = \int_{T \backslash G} f(g) \varphi(g\gamma^z \cdot v_3) dg = \varphi(\gamma^z \cdot v_3) \int_{I_f} f(k_0) dk_0.$$

If we write  $k_0 = \begin{pmatrix} 1 & b_0 \\ c_0 & 1 \end{pmatrix} \in I_f$ , then by separating the variables  $b_0$  and  $c_0$  we obtain

$$\int_{I_f} f(k_0) dk_0 = \begin{cases} |\pi|^{x-y} & , \text{ if } \mu_1 \text{ and } \mu'_2 \text{ are unramified,} \\ 0 & , \text{ otherwise.} \end{cases}$$

From this and from lemma 3.1 we deduce :

**Lemma 4.1.**  $\Phi(f)(\gamma^z \cdot v_3) \neq 0$  if, and only if,  $\mu_1$  and  $\mu'_2$  are both unramified.

## 4.2 From $\Phi$ to $\Psi$ .

Now, we are going to compute  $F = \text{ext}(f)$  as a function on  $G \times G$ . Recall that  $F : G \times G \rightarrow \mathbb{C}$  is a function such that :

- for all  $b_1, b_2 \in B$ ,  $g_1, g_2 \in G$ ,  $F(b_1 g_1, b_2 g_2) = \chi_1(b_1) \chi_2(b_2) \delta^{\frac{1}{2}}(b_1 b_2) F(g_1, g_2)$ ,
- for all  $g \in G$ ,  $F(g, g) = 0$  and  $F(g, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g) = f(g)$ .

Since  $G = BK$ ,  $F$  is uniquely determined by its restriction to  $K \times K$ . Following the notations of paragraph 2.3 put

$$\alpha_i^{-1} = \mu_i(\pi) |\pi|^{\frac{1}{2}} \quad \text{and} \quad \beta_i^{-1} = \mu'_i(\pi) |\pi|^{-\frac{1}{2}}.$$

**Lemma 4.2.** Suppose that  $x - n_3 \geq z \geq y \geq 0$  and  $x - y \geq \max(n_1 - m_1, n_2 - m_2, 1)$ . Then for all  $k_1 = \begin{pmatrix} * & * \\ c_1 & d_2 \end{pmatrix}$  and  $k_2 = \begin{pmatrix} * & * \\ c_2 & d_2 \end{pmatrix}$  in  $K$  we have  $F(k_1, k_2) = 0$  unless

$$d_1 c_2 \neq 0, \quad \frac{c_1}{d_1} \in \pi^x \mathcal{O}^{\mu_1} \quad \text{and} \quad \frac{d_2}{c_2} \in \pi^{-y} \mathcal{O}^{\mu'_2},$$

in which case, if we denote by  $s$  the valuation of  $c_2$ , we have

$$F(k_1, k_2) = \begin{cases} \mu_1 \left( \frac{\det(k_1)}{\pi^{-x} c_1} \right) \mu'_1(d_1) \mu_2 \left( \frac{-\det(k_2)}{\pi^{-s} c_2} \right) \mu'_2(d_2) \left( \frac{\alpha_2}{\beta_2} \right)^s & , \text{ if } \mu_1 \text{ and } \mu'_2 \text{ are ramified ;} \\ \mu'_1(d_1) \mu_2 \left( \frac{-\det(k_2)}{\pi^{-s} c_2} \right) \mu'_2(d_2) \left( \frac{\alpha_2}{\beta_2} \right)^s & , \text{ if } \mu_1 \text{ is unramified and } \mu'_2 \text{ is ramified ;} \\ \mu_1 \left( \frac{\det(k_1)}{\pi^{-x} c_1} \right) \mu'_1(d_1) \mu_2 \left( \frac{-\det(k_2)}{\pi^{-s} c_2} \right) \left( \frac{\alpha_2}{\beta_2} \right)^s & , \text{ if } \mu_1 \text{ is ramified and } \mu'_2 \text{ is unramified ;} \\ \mu'_1(d_1) \mu_2 \left( \frac{-\det(k_2)}{\pi^{-s} c_2} \right) \left( \frac{\alpha_2}{\beta_2} \right)^s & , \text{ if } \mu_1 \text{ and } \mu'_2 \text{ are unramified.} \end{cases}$$

*Proof :* By definition  $F(k_1, k_2) = 0$  unless there exist  $k_0 = \begin{pmatrix} 1 & b_0 \\ c_0 & 1 \end{pmatrix} \in I_f$  such that

$$k_1 k_0^{-1} \in B \quad \text{and} \quad k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in B,$$

in which case

$$F(k_1, k_2) = \chi_1(k_1 k_0^{-1}) \chi_2 \left( k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \delta^{\frac{1}{2}} \left( k_1 k_0^{-1} k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) f(k_0).$$

From  $k_1 k_0^{-1} \in B$ , we deduce that  $c_1 = c_0 d_1$ . From  $k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in B$  we deduce that  $d_2 = b_0 c_2$ . Hence

$$d_1 \in \mathcal{O}^\times, \quad \frac{c_1}{d_1} \in \pi^x \mathcal{O}^{\mu_1}, \quad c_2 \neq 0 \quad \text{and} \quad \frac{d_2}{c_2} \in \pi^{-y} \mathcal{O}^{\mu'_2}.$$

Moreover

$$k_1 k_0^{-1} = \begin{pmatrix} \frac{\det k_1}{d_1 \det k_0} & * \\ 0 & d_1 \end{pmatrix} \quad \text{and} \quad k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{-\det k_2}{c_2 \det k_0} & * \\ 0 & c_2 \end{pmatrix}.$$

Since  $x - y \geq n_1 - m_1$ ,  $x - y \geq n_2 - m_2$  and  $x - y \geq 1$  we have

$$\mu_1(\det k_0) = \mu_2(\det k_0) = 1.$$

Hence

$$F(k_1, k_2) = \mu_1 \left( \frac{\det k_1}{d_1} \right) \mu'_1(d_1) \mu_2 \left( \frac{-\det k_2}{c_2} \right) \mu'_2(c_2) \left| \frac{1}{c_2} \right| f \left( \frac{1}{\frac{c_1}{d_1}} \frac{\frac{d_2}{c_2}}{1} \right).$$

From here and (9) follows the desired formula for  $F$ .

Conversely, if  $k_1$  and  $k_2$  are such that  $\frac{c_1}{d_1} \in \pi^x \mathcal{O}^{\mu_1}$  and  $\frac{d_2}{c_2} \in \pi^{-y} \mathcal{O}^{\mu'_2}$  one can take

$$k_0 = \begin{pmatrix} 1 & d_2 c_2^{-1} \\ c_1 d_1^{-1} & 1 \end{pmatrix}.$$

□

**Remark 4.3.** One can compute  $F$  without the assumption  $x - y \geq \max(n_1 - m_1, n_2 - m_2, 1)$ . However,  $F$  needs not decompose as a product of functions of one variable as in the above lemma.

For example, if  $x = n_3 = 0$  and  $n_1 = n_2$ , then for all  $k_1 \in K$  and  $k_2 \in K$

$$F(k_1, k_2) = \begin{cases} \omega_1 \left( \frac{c_1 d_2 - d_1 c_2}{\det k_2} \right) & , \text{ if } d_1 \in \mathcal{O}^\times, \quad c_2 \in \mathcal{O}^\times \text{ and } c_1 d_2 \neq d_1 c_2 \\ 0 & , \text{ otherwise.} \end{cases}$$

### 4.3 From $\Psi$ to $\ell$

Now, we want to express  $F \in V_1 \otimes V_2$  in terms of the new vectors  $v_1$  and  $v_2$ .

From now on we suppose that  $x, y$  and  $z$  are integers as in theorem 7. We may also suppose that  $x \geq 1$ , because otherwise  $V_1, V_2$  and  $V_3$  are all unramified and this case is covered in Theorem 2. Observe also that if  $y = 0$ , then  $\mu'_2$  is unramified and therefore  $\mathcal{O}^{\mu'_2} = \mathcal{O}$ .

For  $i = 1, 2$ , since  $k_i \in K$ , both  $c_i$  and  $d_i$  are in  $\mathcal{O}$ , and one of them is in  $\mathcal{O}^\times$ . Hence

- $\frac{c_1}{d_1} \in \pi^x \mathcal{O}^\times$  if, and only if  $k_1 \in I_x \setminus I_{x+1}$ ,
- $\frac{c_1}{d_1} \in \pi^x \mathcal{O}$  if, and only if  $k_1 \in I_x$ ,
- $\frac{d_2}{c_2} \in \pi^{-y} \mathcal{O}^\times$  with  $y \geq 1$  if, and only if  $k_2 \in I_y \setminus I_{y+1}$ ,
- $\frac{d_2}{c_2} \in \pi^{-y} \mathcal{O}$  with  $y \geq 0$  if, and only if  $k_2 \in K \setminus I_{y+1}$ .

**Lemma 4.4.** *With the notations of (1),  $F$  is a non-zero multiple of  $v_1^* \otimes v_2^*$ .*

*Proof :* Both  $F$  and  $v_1^* \otimes v_2^*$  are elements in  $\text{Ind}_{B \times B}^{G \times G}(\chi_1 \times \chi_2)$ , hence it is enough to compare their restrictions to  $K \times K$ . By the above discussion together with lemmas 4.2 and 2.5 the two restrictions are supported by

$$\begin{cases} (I_x \setminus I_{x+1}) \times (I_y \setminus I_{y+1}) & , \text{ if } \mu_1 \text{ and } \mu'_2 \text{ are ramified ;} \\ I_x \times (I_y \setminus I_{y+1}) & , \text{ if } \mu_1 \text{ is unramified and } \mu'_2 \text{ is ramified ;} \\ (I_x \setminus I_{x+1}) \times (K \setminus I_{y+1}) & , \text{ if } \mu_1 \text{ is ramified and } \mu'_2 \text{ is unramified ;} \\ I_x \times (K \setminus I_{y+1}) & , \text{ if } \mu_1 \text{ and } \mu'_2 \text{ are unramified.} \end{cases}$$

There are 16 different cases depending on whether each one among  $\mu_1, \mu'_1, \mu_2$  and  $\mu'_2$  is ramified or unramified. Since it is a straightforward verification from lemmas 2.1, 2.2, 2.3 and 2.4, in order to avoid repetitions or cumbersome notations, we will only give the final result :

$$F = \lambda_1 \lambda_2 \mu_2 (-1) \alpha_1^{m_1-x} \alpha_2^{m_2} \beta_2^{-y} (v_1^* \otimes v_2^*) , \text{ where} \quad (10)$$

$$\lambda_i = \begin{cases} \left(1 - \frac{\beta_i}{\alpha_i}\right)^{-1} & , \text{ if } V_i \text{ is unramified,} \\ 1 & , \text{ if } V_i \text{ is ramified.} \end{cases}$$

In all cases  $F$  is a non-zero multiple of  $v_1^* \otimes v_2^*$ .  $\square$

Since by definition  $\ell(F \otimes \bullet) = \Psi(F) = \Phi(f)$ , the above lemma together with lemma 4.1 imply theorem 7.

### 4.4 Proof of Theorems 5 and 6.

We assume henceforth that  $\mu_1$  and  $\mu'_2$  are *both* unramified ( $n_1 - m_1 = m_2 = 0$ ). We put  $y = z = 0$  and  $x = \max(n_1, n_3) \geq 1$ . Since  $\omega_1 \omega_2 \omega_3 = 1$ ,  $\max(n_1, n_3) = \max(n_1, n_2, n_3) \geq 1$ .

Then lemma 4.1 yields :

$$\ell(F \otimes v_3) = \Psi(F)(v_3) = \Phi(f)(v_3) \neq 0. \quad (11)$$

From this and lemma 4.4 we deduce :



**Lemma 4.5.** *We have  $\ell(v_1^* \otimes v_2^* \otimes v_3) \neq 0$  where*

$$v_1^* = \begin{cases} \gamma^{x-n_1} \cdot v_1 & , \text{ if } \mu'_1 \text{ is ramified,} \\ \gamma^x \cdot v_1 - \beta_1 \gamma^{x-1} \cdot v_1 & , \text{ if } \mu'_1 \text{ is unramified.} \end{cases}$$

$$v_2^* = \begin{cases} v_2 & , \text{ if } \mu_2 \text{ is ramified.} \\ v_2 - \alpha_2^{-1} \gamma \cdot v_2 & , \text{ if } \mu_2 \text{ is unramified.} \end{cases}$$

#### 4.4.1 The case of two unramified representations.

Suppose that  $n_1 = n_2 = 0$ , so that  $x = n_3$ . Then lemma 4.5 yields :

$$\ell\left((\gamma^{n_3} \cdot v_1 - \beta_1 \gamma^{n_3-1} \cdot v_1) \otimes (\gamma \cdot v_2 - \alpha_2 v_2) \otimes v_3\right) \neq 0.$$

This expression can be simplified as follows. Consider for  $m \geq 0$  the linear form :

$$\psi_m(\bullet) = \ell(\gamma^m \cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V}_3.$$

As observed in the introduction,  $\psi_m$  is invariant by  $\gamma^m K \gamma^{-m} \cap K = I_m$ , hence vanishes if  $m < n_3 = \text{cond}(\widetilde{V}_3)$ . Therefore, for  $n_3 \geq 2$  :

$$\begin{aligned} & \ell\left((\gamma^{n_3} \cdot v_1 - \beta_1 \gamma^{n_3-1} \cdot v_1) \otimes (\gamma \cdot v_2 - \alpha_2 v_2) \otimes v_3\right) \\ &= -\alpha_2 \psi_{n_3}(v_3) + \beta_1 \alpha_2 \psi_{n_3-1}(v_3) + \psi_{n_3-1}(\gamma^{-1} \cdot v_3) - \beta_1 \psi_{n_3-2}(\gamma^{-1} \cdot v_3) \\ &= -\alpha_2 \psi_{n_3}(v_3) \\ &= -\alpha_2 \ell(\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3) \neq 0. \end{aligned}$$

If  $n_3 = 1$ , only the two terms in the middle vanish and we obtain

$$\alpha_2 \ell(\gamma \cdot v_1 \otimes v_2 \otimes v_3) + \beta_1 \ell(v_1 \otimes \gamma \cdot v_2 \otimes v_3) \neq 0.$$

Put  $g = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$ . Then  $g\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K$  and  $\gamma^{-1}g = \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix} \in \pi K$ . Hence :

$$\begin{aligned} \beta_1 \ell(v_1 \otimes \gamma \cdot v_2 \otimes v_3) &= \beta_1 \ell(\gamma \gamma^{-1} g \cdot v_1 \otimes g \gamma \cdot v_2 \otimes g \cdot v_3) \\ &= \beta_1 \omega_1(\pi) \ell(\gamma \cdot v_1 \otimes v_2 \otimes g \cdot v_3) \\ &= \alpha_1^{-1} \ell(\gamma \cdot v_1 \otimes v_2 \otimes g \cdot v_3). \end{aligned}$$

Therefore

$$\ell(\gamma \cdot v_1 \otimes v_2 \otimes (g \cdot v_3 + \alpha_1 \alpha_2 v_3)) \neq 0,$$

in particular

$$\Psi(\gamma \cdot v_1 \otimes v_2) \neq 0.$$

By the same argument as in paragraph 3.2.4 we conclude that

$$\ell(\gamma \cdot v_1 \otimes v_2 \otimes v_3) = \Psi(\gamma \cdot v_1 \otimes v_2)(v_3) \neq 0.$$

Hence, if  $n_3 \geq 1$ ,  $\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3$  is a test vector. This completes the proof of Theorem 5.

#### 4.4.2 The case of two ramified principal series.

Suppose that  $V_1$  and  $V_2$  are both ramified ( $m_1 > 0$ ,  $n_1 - m_1 = 0$ ,  $m_2 = 0$ ,  $n_2 > 0$ ) and put  $n = x - n_1 = \max(n_2 - n_1, n_3 - n_1)$ . Then lemma 4.5 yields :

$$\ell(\gamma^n \cdot v_1 \otimes v_2 \otimes v_3) \neq 0,$$

hence  $\gamma^n \cdot v_1 \otimes v_2 \otimes v_3$  is a test vector.

#### 4.4.3 The case where $V_1$ is unramified and $V_2$ is ramified.

Suppose that  $n_1 = 0$ , but  $n_2 > 0$ . Then  $x = n_3 \geq n_2$  and lemma 4.5 yields :

$$\ell\left((\gamma^{n_3} \cdot v_1 - \beta_1 \gamma^{n_3-1} \cdot v_1) \otimes v_2 \otimes v_3\right) \neq 0.$$

If  $n_2 < n_3$ , then

$$\gamma^{n_3-1} K \gamma^{1-n_3} \cap I_{n_2} \supset I_{n_3-1},$$

and therefore

$$\ell(\gamma^{n_3-1} \cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V}_3^{I_{n_3-1}, \omega_3^{-1}} = \{0\}.$$

Hence

$$\ell(\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3) \neq 0,$$

that is  $\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3$  is a test vector.

If  $n_2 = n_3$ , the condition on the central character forces  $V_3$  and  $\omega_3$  to have the same conductor. Hence  $V_3$  is also a principal series. In this case we do not see *a priori* a reason for either  $\ell(\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes v_3)$  or  $\ell(\gamma^{n_3-1} \cdot v_1 \otimes v_2 \otimes v_3)$  to vanish. But we can notice that the two linear forms

$$\ell(\gamma^{n_3} \cdot v_1 \otimes v_2 \otimes \bullet) \quad \text{and} \quad \ell(\gamma^{n_3-1} \cdot v_1 \otimes v_2 \otimes \bullet)$$

belong both to the new line  $\widetilde{V}_3^{I_n, \omega_3^{-1}}$  of  $\widetilde{V}_3$ , hence they are proportionals.

#### 4.4.4 The case where $V_1$ is ramified and $V_2$ is unramified.

Suppose that  $n_1 > 0$  and  $n_2 = 0$ . Then  $x = n_3 \geq n_1$  and lemma 4.5 yields :

$$\ell\left(\gamma^{n_3-n_1} \cdot v_1 \otimes (\gamma \cdot v_2 - \alpha_2 v_2) \otimes v_3\right) \neq 0.$$

If  $n_1 < n_3$ , then

$$\ell(\gamma^{n_3-n_1-1} \cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V}_3^{I_{n_3-1}, \omega_3^{-1}} = \{0\}.$$

Then

$$\ell(\gamma^{n_3-n_1} \cdot v_1 \otimes \gamma \cdot v_2 \otimes v_3) = \ell(\gamma^{n_3-n_1-1} \cdot v_1 \otimes v_2 \otimes \gamma^{-1} \cdot v_3) = 0.$$

Hence

$$\ell(\gamma^{n_3-n_1} \cdot v_1 \otimes v_2 \otimes v_3) \neq 0,$$

that is  $\gamma^{n_3-n_1} \cdot v_1 \otimes v_2 \otimes v_3$  is a test vector.

If  $n_1 = n_3$ , the condition on the central character forces  $V_3$  to be also a principal series. In this case we do not see *a priori* a reason for either  $\ell(v_1 \otimes v_2 \otimes v_3)$  or  $\ell(v_1 \otimes \gamma \cdot v_2 \otimes v_3)$  to vanish. But we can once again notice that the two linear forms

$$\ell(v_1 \otimes v_2 \otimes \bullet) \quad \text{and} \quad \ell(v_1 \otimes \gamma \cdot v_2 \otimes \bullet)$$

belong to the line generated by a new vector in  $\widetilde{V}_3$ , hence are proportionals.

The proof of Theorem 6 is now complete.

## References

- [B-H] Colin J. Busnell and Guy Henniart, *The local Langlands conjecture for  $GL(2)$* . Springer Series : Grundlehren der mathematischen Wissenschaften, Vol. **335** (2007) .
- [B-Z] Joseph Bernstein and Andrei Zelevinsky, *Representations of the group  $GL(n, F)$  where  $F$  is a non-archimedian local field*. Russian Mathematical Surveys **31:3** (1976), 1-68.
- [B-R 1] Joseph Bernstein and Andre Reznikov, *Estimates of automorphic functions*. Moscow Mathematic Journal **4**, no.1 (2004), 19-37.
- [B-R 2] Joseph Bernstein and Andre Reznikov, *Periods, subconvexity and representation theory*. Journal of differential geometry **70** (2005), 129-142.
- [C] , William Casselman *On some Results of Atkin and Lehner*. Mathematische Annalen **201** (1973), 301-314.
- [G-P] Benedict H.Gross and Diprenda Prasad, *Test Vectors for Linear forms*. Mathematische Annalen **291** (1991), 343-355.
- [H-S] Michael Harris and Anthony Scholl, *A note on trilinear forms for reducible representations and Beilinson conjectures*. Journal of the European Mathematical Society **2001**, **1** (2001), 93-104.
- [M-V] Philippe Michel and Akshay Venkatesh, *Equidistribution,  $L$ -functions and Ergodic theory : on some problem of Yu. V. Linnik*. In *International Congress of Mathematicians 2006, Madrid*, Volume II, 421-458. European Mathematical Society, Zurich.
- [P] Diprenda Prasad, *Trilinear forms for representations of  $GL(2)$  and local  $\varepsilon$ -factors*. Compositio Mathematica **75** (1990), 1-46.
- [T] J. Tunnell, *Local  $\varepsilon$ -factors and characters of  $GL(2)$* . American Journal of Mathematics **105** (1983), 1277-1308.
- [V] Akshay Venkatesh, *Sparse equidistribution problems, period bounds, and subconvexity*. Preprint (2005).
- [W] Jean-Loup Waldspurger, *Sur les valeurs de certaines fonctions  $L$  automorphes en leur centre de symétrie*. Compositio Mathematica **54** (1985), 173-242.